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Michel Waldschmidt

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# On the numbers $e^e, e^{e^2}$ and $e^{\pi^2}$

by

**Michel WALDSCHMIDT**

**Abstract.** We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple  $(e, e^e, e^{e^2})$  and the second one for  $(\pi, e, e^{\pi^2})$ . We deduce from these measures two transcendence results which had been proved in the early 70's by W.D. Brownawell and the author.

## Introduction

In 1949, A.O. Gel'fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers  $2^{\sqrt[3]{2}}$  and  $2^{\sqrt[3]{4}}$  are algebraically independent. At the same time, he proved that one at least of the three numbers  $e^e, e^{e^2}, e^{e^3}$  is transcendental (see[G] Chap. III, §4).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers  $e^e, e^{e^2}$  is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M. Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

- (i) The numbers  $e$  and  $\pi$  are algebraically independent
- (ii) The number  $e^{\pi^2}$  is transcendental

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:

(?) Two at least of the three numbers  $e$ ,  $e^e$ ,  $e^{e^2}$  are algebraically independent.

(?) Two at least of the three numbers  $\pi$ ,  $e$ ,  $e^{\pi^2}$  are algebraically independent.

Further conjectures are as follows:

(?) Each of the numbers  $e^e$ ,  $e^{e^2}$ ,  $e^{\pi^2}$  is transcendental

(?) The numbers  $e$  and  $\pi$  are algebraically independent.

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

## 1. Heights

Let  $\gamma$  be a complex algebraic number. The *minimal polynomial* of  $\gamma$  over  $\mathbb{Z}$  is the unique polynomial

$$f(X) = a_0X^d + a_1X^{d-1} + \dots + a_d \in \mathbb{Z}[X]$$

which vanishes at the point  $\gamma$ , is irreducible in the factorial ring  $\mathbb{Z}[X]$  and has leading coefficient  $a_0 > 0$ . The integer  $d = \deg f$  is the *degree* of  $\gamma$ , denoted by  $[Q(\gamma) : \mathbb{Q}]$ . The *usual height*  $H(\gamma)$  of  $\gamma$  is defined by

$$H(\gamma) = \max\{|a_0|, \dots, |a_d|\}.$$

It will be convenient to use also the so-called *Mahler's measure* of  $\gamma$ , which can be defined in three equivalent ways. The first one is

$$M(\gamma) = \exp \left( \int_0^1 \log |f(e^{2i\pi t})| dt \right).$$

For the second one, let  $\gamma_1, \dots, \gamma_d$  denote the complex roots of  $f$ , so that

$$f(X) = a_0 \prod_{i=1}^d (X - \gamma_i).$$

Then, according to Jensen's formula, we have

$$M(\gamma) = |a_0| \prod_{i=1}^d \max\{1, |\gamma_i|\}.$$

For the third one, let  $K$  be a number field (that is a subfield of  $\mathbb{C}$  which is a  $\mathbb{Q}$ -vector space of finite dimension  $[K:\mathbb{Q}]$ ) containing  $\gamma$ , and let  $M_K$  be the set of (normalized) absolute

values of  $K$ . Then

$$M(\gamma) = \prod_{v \in M_K} \max\{1, |\gamma|_v\}^{[K_v:\mathbb{Q}]}$$

where  $K_v$  is the completion of  $K$  for the absolute value  $v$  and  $Q_v$  the topological closure of  $\mathbb{Q}$  in  $K_v$  and  $[K_v : \mathbb{Q}_v]$  the local degree.

Mahler's measure is related to the usual height by

$$2^{-d}H(\gamma) \leq M(\gamma) \leq \sqrt{d+1} \cdot H(\gamma).$$

From this point of view it does not make too much difference to use  $H$  or  $M$ , but one should be careful that  $d$  denotes the exact degree of  $\gamma$ , not an upper bound. We shall deal below with algebraic numbers of degree  $d$  bounded by some parameter  $D$ .

*Definition.* For an algebraic number  $\gamma$  of degree  $d$  and Mahler's measure  $M(\gamma)$ , we define the *absolute logarithmic height*  $h(\gamma)$  by

$$h(\gamma) = \frac{1}{d} \log M(\gamma).$$

## 2. Simultaneous Approximation

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

### 2.1. Simultaneous Approximation to $e, e^e$ and $e^{e^2}$

**Theorem 1.** *There exists a positive absolute constant  $c_1$  such that, if  $\gamma_0, \gamma_1, \gamma_2$  are algebraic numbers in a field of degree  $D$ , then*

$$|e - \gamma_0| + |e^e - \gamma_1| + |e^{e^2} - \gamma_2| > \exp\{-c_1 D^2 (h_0 + h_1 + h_2)^{1/2} (h_1 + h_2)^{1/2} (h_0 + \log D)(\log D)^{-1}\}$$

where  $h_i = \max\{e, h(\gamma_i)\}$  ( $i = 0, 1, 2$ ).

## 2.2. Simultaneous Approximation to $\pi, e$ and $e^{\pi^2}$

**Theorem 2.** *There exists a positive absolute constant  $c_2$  such that, if  $\gamma_0, \gamma_1, \gamma_2$  are algebraic numbers in a field of degree  $D$ , then*

$$|\pi - \gamma_0| + |e^e - \gamma_1| + |e^{e^2} - \gamma_2| > \exp\{-c_2 D^2 (h_0 + \log(Dh_1 h_2))^{\frac{1}{2}} h_1^{\frac{1}{2}} h_2^{\frac{1}{2}} (\log D)^{-1}\}$$

where  $h_i = \max\{e, h(\gamma_i)\}$  ( $i = 0, 1, 2$ ).

## 3. Transcendence Criterion

### 3.1 Algebraic Approximations to a Given Transcendental Number

The following result is Théorème 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

**Theorem 3.** *Let  $\theta \in \mathbb{C}$  be a complex number. The two following conditions are equivalent:*

- (i) *the number  $\theta$  is transcendental.*
- (ii) *For any real number  $h \geq 10^7$ , there are infinitely many integers  $d \geq 1$  for which there exists an algebraic number  $\gamma$  of degree  $d$  and absolute logarithmic height  $h(\gamma) \leq h$  which satisfies*

$$0 < |\theta - \gamma| \leq \exp(-10^{-7} h d^2).$$

Notice that the proof of (ii)  $\Rightarrow$  (i) is an easy consequence of Liouville's inequality.

### 3.2. Application to $e^e$ and $e^{e^2}$

**Corollary to Theorem 1.** *One at least of the two numbers  $e^e, e^{e^2}$  is transcendental.*

*Proof of the corollary.* Assume that the two numbers  $e^e, e^{e^2}$  are algebraic, say  $\gamma_1$  and  $\gamma_2$ . Then, according to Theorem 1, there exists a constant  $c_3 > 1$  such that, for any algebraic number  $\gamma$  of degree  $\leq D$  and height  $h(\gamma) \leq h$  with  $h \geq e$ ,

$$|e - \gamma| > \exp\{-c_3 D^2 h^{1/2} (h + \log D) (\log D)^{-1}\}.$$

We now use Theorem 3 for  $\theta = e$  with  $h = 10^{15} c_3^2$  and derive a contradiction.

## 4. Algebraic Independence

### 4.1 Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théorème 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

**Corollary to Theorem 3.** *Let  $\theta_1, \dots, \theta_m$  be complex numbers such that the field  $\mathbb{Q}(\theta_1, \dots, \theta_m)$  has transcendence degree 1 over  $\mathbb{Q}$ . There exists a constant  $c > 0$  such that, for any real number  $h \geq c$ , there are infinitely many integers  $D$  for which there exists a tuple  $(\gamma_1, \dots, \gamma_m)$  of algebraic numbers satisfying*

$$[\mathbb{Q}(\gamma_1, \dots, \gamma_m) : \mathbb{Q}] \leq D, \quad \max_{1 \leq i \leq m} h(\gamma_i) \leq h$$

and

$$\max_{1 \leq i \leq m} \{|\theta_i - \gamma_i|\} \leq \exp(-c^{-1}hD^2).$$

### 4.2. Application to $\pi, e$ and $e^{\pi^2}$

**Corollary to Theorem 2.** *One at least of the two following statements is true:*

- (i) *The numbers  $e$  and  $\pi$  are algebraically independent.*
- (ii) *The number  $e^{\pi^2}$  is transcendental.*

*Remark.* This corollary can be stated in an equivalent way as follows:

For any non constant polynomial  $P \in \mathbb{Z}[X]$ , the complex number

$$e^{\pi^2} + iP(e, \pi)$$

is transcendental.

The idea behind this remark originates in [R].

*Proof of the Corollary.* Assume that the number  $e^{\pi^2}$  is algebraic. Theorem 2 with  $\gamma_2 = e^{\pi^2}$  shows that there exists a constant  $c_4 > 0$  such that, for any pair  $(\gamma_0, \gamma_1)$  of algebraic numbers, if we set

$$D = [\mathbb{Q}(\gamma_0, \gamma_1) : \mathbb{Q}] \quad \text{and} \quad h = \max\{e, h(\gamma_0), h(\gamma_1)\},$$

then

$$|\pi - \gamma_0| + |e - \gamma_1| > \exp\{-c_4 D^2 (h + \log D)^{1/2} h^{1/2} (\log D)^{-1}\}$$

Therefore we deduce from the Corollary to Theorem 3 that the field  $Q(\pi, e)$  has transcendence degree 2.

## 5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p. 30: (The results of this section are based on the conjecture to be stated).

**Schanuel's Conjecture.** *Let  $x_1, \dots, x_n$  be  $Q$ -linearly independent complex numbers. Then, among the  $2n$  numbers*

$$x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n},$$

*at least  $n$  are algebraically independent.*

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) *The 7 numbers*

$$e, \pi, e^e, e^{e^2}, e^{\pi^2}, 2^{\sqrt[3]{2}}, 2^{\sqrt[3]{2}}$$

*are algebraically independent.*

We shall use Schanuel's conjecture twice. We start with the numbers  $1, \log 2$ , and  $i\pi$  which are linearly independent over  $Q$  because  $\log 2$  is irrational. Therefore, according to Schanuel's conjecture, three at least of the numbers

$$1, \log 2, i\pi, e, 2, -1$$

are algebraically independent. This means that the three numbers  $\log 2, \pi$  and  $e$  are algebraically independent. Therefore the 8 numbers

$$1, i\pi, \pi^2, e, e^2, \log 2, 2^{\frac{1}{3}} \log 2, 4^{\frac{1}{3}} \log 2$$

are  $Q$ -linearly independent. Again, Schanuel's conjecture implies that 8 at least of the numbers

$$1, i\pi, \pi^2, e, e^2, \log^2, 2^{\frac{1}{3}} \log 2, 4^{\frac{1}{3}} \log 2, e, -1, e^{\pi^2}, e^e, e^{e^2}, 2, 2^{\sqrt[3]{2}}, 2^{\sqrt[3]{4}}$$

are algebraically independent, and this means that the 8 numbers

$$e, \pi, e^e, e^{e^2}, e^{\pi^2}, 2^{\sqrt[3]{2}}, 2^{\sqrt[3]{4}}, \log 2$$

are algebraically independent.

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Michel Waldschmidt  
 Université P.et M. Curie (Paris VI)  
 Institut Mathématique de Jussieu  
 Problèmes Diophantiens, Case 247  
 4, Place Jussieu  
 F-75252, Paris CEDEX05  
 FRANCE.

E-mail: miw@math.jussieu.fr

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